

# On Some New Discrete Generalizations of Gronwall's Inequality

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This paper derives new discrete generalizations of the Gronwall-Bellman integral inequality. These generalizations should have wide application in the study of finite difference equations and numerical analysis. The main result (Theorem 3) concerns a very general form of linear Bellman-type discrete inequalities in one independent variable. It is a discrete analogue of an integral inequality obtained by the present author in [*J. Math. Anal. Appl.* **103** (1984), 184-197, Theorem 4] and it extends many discrete inequalities of Agarwal and Thandapani, Pachpatte, and Sugiyama. Two nonlinear extensions of Theorem 3 are also established here. © 1988 Academic Press, Inc.

## INTRODUCTION

A very useful technique in the study of many problems concerning the behavior of solutions of discrete time system is to use recurrent inequalities involving sequences of real numbers, which may be considered as a discrete analogue of the Gronwall-Bellman integral inequality [2] or its generalizations. During the last few years the area of applications of discrete inequalities has greatly expanded, and now encompasses not only many problems in the theory of finite difference equations and numerical analysis but also some questions of physics, technology, economics, and biological sciences. The discovery of new discrete inequalities and their new applications has attracted much interest from many authors (see, for example, [1, 3-19, 21, 22]).

We recall here a few notations and definitions which are commonly used in the literature. Let  $N$  be the infinite countable set consisting of the numbers  $n_0, n_0 + 1, \dots, n_0 + k, \dots$ , where  $n_0 \geq 0$  is a given integer. Define by  $\Delta$  the difference operator such that  $\Delta y(n) = y(n+1) - y(n)$ , and for any real-valued function  $f(n)$  on  $N$  we define

$$\sum_{s=n_0}^{n-1} f(s) = \sum_{p=0}^{k-1} f(n_0 + p), \quad \prod_{s=n_0}^{n-1} f(s) = \prod_{p=0}^{k-1} f(n_0 + p), \quad n = n_0 + k.$$

For convenience we take an empty sum to be zero and an empty product to be one.

## 1. BASIC CASE

**THEOREM 1.** *Let  $x(n), p(n)$  be real-valued nonnegative functions defined on  $N$  with  $p$  nondecreasing on  $N$ , and for  $j = 1, 2, \dots, m$  let  $f_j(n, s)$  be real-valued nonnegative functions defined on  $N \times N$ , which are nondecreasing in  $n$  for every  $s \in N$  fixed. Suppose that the discrete inequality*

$$\begin{aligned} x(n) \leq p(n) + \sum_{s_1=n_0}^{n-1} f_1(n, s_1) \sum_{s_2=n_0}^{s_1-1} f_2(s_1, s_2) \cdots \\ \times \sum_{s_m=n_0}^{s_{m-1}-1} f_m(s_{m-1}, s_m) \times (s_m) \end{aligned} \quad (1.1)$$

holds for all  $n \in N$ . Then we have

$$x(n) \leq p(n) W_m(n), \quad n \in N, \quad (1.2)$$

where  $W_m(n) = V_m(n, n)$ , and  $V_m(r, q)$  is defined by

$$\begin{aligned} V_1(r, q) &= \prod_{s=n_0}^{q-1} \left\{ 1 + \sum_{j=1}^m f_j(r, s) \right\}, \\ V_k(r, q) &= \prod_{s=n_0}^{q-1} g_{m-k+1}(r, s) + \sum_{s=n_0}^{q-1} f_{m-k+1}(r, s) V_{k-1}(r, s) \\ &\times \prod_{t=s+1}^{q-1} g_{m-k+1}(r, t), \quad k = 2, 3, \dots, m, \end{aligned} \quad (1.3)$$

where

$$g_h(r, q) = \begin{cases} 1 + \sum_{j=1}^{h-1} f_j(r, q) - f_h(r, q), \\ \quad \text{if this expression} \geq 0 \text{ on } N \times N, \\ 1 + \sum_{j=1}^{h-1} f_j(r, q), \\ \quad \text{otherwise} \quad \text{for } h = 1, 2, \dots, m-1. \end{cases} \quad (1.4)$$

*Proof.* For every  $c \in N$  fixed and any real-valued nonnegative function  $v(n)$  on  $N$ , we define

$$\begin{aligned}
I_h(c, n; v) &= f_h(c, n) \sum_{s_{h+1}=n_0}^{n-1} f_{h+1}(c, s_{h+1}) \sum_{s_{h+2}=n_0}^{s_{h+1}-1} f_{h+2}(c, s_{h+2}) \\
&\quad \times \sum_{s_m=n_0}^{s_{m-1}-1} f_m(c, s_m) v(s_m), \quad h = 1, 2, \dots, m-1 \\
I_m(c, n; v) &= f_m(c, n) v(n).
\end{aligned} \tag{1.5}$$

We observe from (1.5) that

$$I_{k-1}(c, n; v) = f_{k-1}(c, n) \sum_{s_k=n_0}^{n-1} I_k(c, s_k; v), \quad k = 2, 3, \dots, m$$

and all  $I_j(c, n; v)$  are nondecreasing in  $v$  (that is, if  $0 \leq x(n) \leq y(n)$  for  $n \in N$ , then  $I_j(c, n; x) \leq I_j(c, n; y)$  for  $n \in N, j = 1, 2, \dots, m$ ).

Clearly, the estimate for  $x(n)$  in (1.2) holds when  $n = n_0$ , since it reduces to the known relation  $x(n_0) \leq p(n_0)$  of (1.1). Now, fixing an arbitrary integer  $n_1 (> n_0)$  from  $N$ , then we get from (1.1)

$$x(n) \leq p(n_1) + \sum_{s_1=n_0}^{n-1} I_1(n_1, s_1; x), \quad \text{for } n \in \{n_0; n_1\}, \tag{1.6}$$

where  $\{n_0; n_1\}$  denotes the finite set consisting of the integers  $n_0, n_0 + 1, \dots, n_1$ . To derive the upper bound on  $x(n)$  from (1.6), we define

$$\begin{aligned}
K_1(n) &= p(n_1) + \sum_{s_1=n_0}^{n-1} I_1(n_1, s_1; x), \\
K_k(n) &= K_{k-1}(n) + \sum_{s_k=n_0}^{n-1} I_k(n_1, s_k; K_{k-1}), \\
&\quad \text{for } k = 2, 3, \dots, m, \text{ and } n \in \{n_0; n_1\}.
\end{aligned} \tag{1.7}$$

Obviously, here we have

$$\begin{aligned}
0 &\leq p(n_1) = K_j(n_0), & j &= 1, 2, \dots, m, \\
0 &\leq x(n) \leq K_1(n) \leq K_2(n) \leq \dots \leq K_m(n), & n &\in \{n_0; n_1\}.
\end{aligned} \tag{1.8}$$

We notice that the following discrete inequalities for  $K_j(n)$  can be established by induction:

$$\begin{aligned}
&\Delta k_h(n) + f_h(n_1, n) K_h(n) \\
&\leq \sum_{j=1}^{h-1} f_j(n_1, n) K_h(n) + f_h(n_1, n) K_{h+1}(n), \\
&\quad \text{for } h = 1, 2, \dots, m-1; n \in \{n_0; n_1 - 1\}.
\end{aligned} \tag{1.9}$$

In fact, noting that the  $I_j(n_1, n; v)$  are nondecreasing in  $v$ , we can use (1.8) to derive from the first equality of (1.7) that

$$\Delta K_1(n) = I_1(n_1, n; x) \leq I_1(n_1, n; K_1), \quad n \in \{n_0; n_1\}.$$

Adding  $f_1(n_1, n) K_1(n)$  to both sides of the above inequality, we obtain

$$\begin{aligned} \Delta K_1(n) + f_1(n_1, n) K_1(n) &\leq f_1(n_1, n) K_1(n) + I_1(n_1, n; K_1) \\ &= f_1(n_1, n) \left\{ K_1(n) + \sum_{s_2=n_0}^{n-1} I_2(n_1, s_2; K_1) \right\} \\ &= f_1(n_1, n) K_2(n), \quad \text{for } n \in \{n_0; n_1 - 1\}. \end{aligned}$$

The last inequality establishes the validity of (1.9) for  $h=1$ . We now suppose that (1.9) holds for  $h=i$ , where  $1 \leq i \leq m-2$ . Then we obtain from (1.7) that

$$\begin{aligned} \Delta K_{i+1}(n) &= \Delta K_i(n) + I_{i+1}(n_1, n; K_i) \\ &\leq \sum_{j=1}^{i-1} f_j(n_1, n) K_i(n) + f_i(n_1, n) K_{i+1}(n) \\ &\quad + I_{i+1}(n_1, n; K_i), \quad \text{for } n \in \{n_0; n_1 - 1\}. \end{aligned}$$

Adding  $f_{i+1}(n_1, n) K_{i+1}(n)$  to both sides of the last inequality and using (1.8) and the monotonicity of  $I_j(n_1, n; v)$  in  $v$ , we get

$$\begin{aligned} \Delta K_{i+1}(n) + f_{i+1}(n_1, n) K_{i+1}(n) &\leq \sum_{j=1}^i f_j(n_1, n) K_{i+1}(n) + f_{i+1}(n_1, n) K_{i+1}(n) + I_{i+1}(n_1, n; K_i) \\ &\leq \sum_{j=1}^i f_j(n_1, n) K_{i+1}(n) + f_{i+1}(n_1, n) \\ &\quad \times \left\{ K_{i+1}(n) + \sum_{s_{i+2}=n_0}^{n-1} I_{i+2}(n_1, s_{i+2}; K_{i+1}) \right\}, \quad n \in \{n_0; n_1 - 1\}. \end{aligned}$$

In view of (1.7), the proof of (1.9) is complete.

Next, we shall derive the upper bound on  $x(n)$  from the relations (1.7), (1.8), and (1.9). We observe from the last equality in (1.7) that

$$\begin{aligned} \Delta K_m(n) &= \Delta K_{m-1}(n) + I_m(n_1, n; K_{m-1}) \\ &\leq \sum_{j=1}^{m-2} f_j(n_1, n) K_{m-1}(n) + f_{m-1}(n_1, n) K_m(n) + f_m(n_1, n) K_{m-1}(n) \\ &\leq \sum_{j=1}^m f_j(n_1, n) K_m(n), \quad \text{for } n \in \{n_0; n_1 - 1\} \end{aligned}$$

by (1.8) and the definition of  $I_m(n_1, n; v)$ . Substituting  $n = n_0, n_0 + 1, \dots, n_1 - 1$  in the last inequality, then we have

$$\begin{aligned} K_m(n) &\leq K_m(n_0) \prod_{s=n_0}^{n-1} \left\{ 1 + \sum_{j=1}^m f_j(n_1, s) \right\} \\ &\equiv p(n_1) V_1(n_1, n), \quad n \in \{n_0; n_1 - 1\}, \end{aligned}$$

where  $V_1(n_1, n)$  is given by (1.3). Now substituting this bound for  $K_m(n)$  in that inequality of (1.9) with  $h = m - 1$ , we get

$$\begin{aligned} \Delta K_{m-1}(n) + f_{m-1}(n_1, n) K_{m-1}(n) \\ \leq \sum_{j=1}^{m-2} f_j(n_1, n) K_{m-1}(n) + f_{m-1}(n_1, n) p(n_1) V_1(n_1, n), \end{aligned}$$

i.e.,

$$K_{m-1}(n+1) \leq g_{m-1}(n_1, n) K_{m-1}(n) + f_{m-1}(n_1, n) p(n_1) V_1(n_1, n), \quad (1.10)$$

for  $n \in \{n_0; n_1 - 1\}$ , where  $g_{m-1}(n_1, n)$  is given by (1.4) with  $h = m - 1$ . Substituting in (1.10) the numbers  $n = n_0, n_0 + 1, \dots, n_1 - 1$  or, more precisely, by using an easy inductive argument, we obtain that

$$K_{m-1}(n) \leq p(n_1) V_2(n_1, n), \quad n \in \{n_0 + 1; n_1\}, \quad (1.11)$$

where  $V_2(n_1, n)$  is given by (1.3). Using this bound for  $K_{m-1}(n)$  in the inequality of (1.9) with  $h = m - 2$ , we get

$$K_{m-2}(n+1) \leq g_{m-2}(n_1, n) K_{m-2}(n) + f_{m-2}(n_1, n) p(n_1) V_2(n_1, n), \quad (1.12)$$

for  $n \in \{n_0; n_1 - 1\}$ , where  $g_{m-2}(n_1, n)$  is given by (1.4). By repeating the same argument as used above from (1.10) to (1.11), we have

$$K_{m-2}(n) \leq p(n_1) V_3(n_1, n), \quad n \in \{n_0 + 1; n_1\}. \quad (1.13)$$

Continuing in this way, after  $m - 1$  applications of the same argument, we derive

$$K_1(n) \leq p(n_1) V_m(n_1, n), \quad n \in \{n_0 + 1; n_1\}, \quad (1.14)$$

where  $V_m(n_1, n)$  is defined by (1.4). Now take  $n = n_1$  in (1.14). The proof of Theorem 1 is complete.

We note that if all hypotheses of the last theorem are satisfied except the monotonicity of  $p(n)$ , then we may replace  $p(n)$  by the monotonic function  $\bar{P}(n) = \max\{p(n_0), p(n_0 + 1), \dots, p(n)\}$ , and then apply Theorem 1.

*Remark 1.* Theorem 1 above extends a known discrete inequality due to Sugiyama [2]. A similar result for (1.1) (when  $m=3$  and all functions  $f_j(n, s)$  are independent of  $n$ ) can be found in Pachpatte [12, Theorem 6]. We remark that the additional assumptions  $1 - f_1(n) \geq 0$  and  $1 + f_1(n) - f_2(n) \geq 0$  for  $n \in N$  were also required in [12].

## 2. MORE GENERAL CASE

In the next result we define

$$J_{im}(n; v) = \sum_{s_1=n_0}^{n-1} f_{i1}(n, s_1) \\ \times \sum_{s_2=n_0}^{s_1-1} f_{i2}(s_1, s_2) \cdots \sum_{s_m=n_0}^{s_{m-1}-1} f_{im}(s_{m-1}, s_m) v(s_m).$$

**THEOREM 2.** Let  $x(n), p(n)$  be the same as in Theorem 1 and let  $f_{ik}(n, s)$  be real-valued nonnegative functions on  $N \times N$ , nondecreasing in  $n$  for every fixed  $s \in N$  (here  $i = 1, 2, \dots, r, k = 1, 2, \dots, m$ ). Suppose that the discrete inequality

$$x(n) \leq p(n) + \sum_{i=1}^r J_{im}(n; x) \quad (2.1)$$

holds for all  $n \in N$ . Then we have

$$x(n) \leq p(n) \prod_{i=1}^r U^{(i)}(n), \quad n \in N, \quad (2.2)$$

where  $U^{(i)}(n) = G_m^{(i)}(n, n)$ , and here  $G_m^{(i)}(r, n)$  are given (in the increasing order of the index  $i$ ) by

$$G_1^{(i)}(r, n) = \prod_{s=n_0}^{n-1} \left\{ 1 + \sum_{k=1}^m f_{ik}(r, s) \right\}, \quad (2.3) \\ G_j^{(i)}(r, n) = \prod_{s=n_0}^{n-1} \theta_{i,m-j+1}(r, s) + \sum_{s=n_0}^{n-1} F_{i,m-j+1}(r, s) \\ \times G_{j-1}^{(i)}(r, s) \prod_{t=s+1}^{n-1} \theta_{i,m-j+1}(r, t),$$

for  $i = 1, 2, \dots, r, j = 2, 3, \dots, m$ , where

$$\theta_{ih}(r, n) = \begin{cases} \psi_{ih}(r, n) - F_{ih}(r, n), & \text{if this expression} \geq 0 \text{ on } N \times N, \\ \psi_{ih}(r, n), & \text{otherwise,} \end{cases} \quad (2.4)$$

$$\psi_{ih}(r, n) = 1 + \sum_{j=1}^{h-1} F_{ij}(r, n), \quad h = 1, 2, \dots, m-1, \quad (2.5)$$

and

$$F_{in}(n, s) = f_{in}(n, s) \prod_{q=1}^{i-1} U^{(q)}(n), \quad F_{ij}(n, s) = f_{ij}(n, s),$$

for  $i = 1, 2, \dots, r, j = 2, 3, \dots, m.$  (2.6)

*Proof.* Rewrite the inequality (2.1) as

$$x(n) \leq A_1(n) + J_{1m}(n; x), \quad n \in N,$$

$$A_1(n) = p(n) + \sum_{i=2}^r J_{im}(n; x). \quad (2.7)$$

Obviously  $A_1(n)$  is nonnegative and nondecreasing on  $N$ , so by Theorem 1 we obtain from (2.7) that

$$x(n) \leq A_1(n) U^{(1)}(n), \quad n \in N, \quad (2.8)$$

where  $U^{(1)}(n) = G_m^{(1)}(n, n)$  and  $G_m^{(1)}(r, n)$  is given by (2.3)–(2.6) with  $i = 1$ . The last inequality can be rewritten as

$$x(n) \leq A_2(n) + J_{2m}^*(n; x), \quad n \in N$$

$$A_2(n) = U^{(1)}(n) \left\{ p(n) + \sum_{i=3}^r J_{im}(n; x) \right\}, \quad (2.9)$$

where  $J_{2m}^*(n; x)$  is obtained from  $J_{2m}(n; x)$  by changing the  $f_{21}(n, s)$  to the function  $U^{(1)}(n)f_{21}(n, s)$ . Now, an application of Theorem 1 to (2.9) yields

$$x(n) \leq \prod_{q=1}^2 U^{(q)}(n) \left\{ p(n) + \sum_{i=3}^r J_{im}(n; x) \right\}, \quad n \in N, \quad (2.10)$$

where  $U^{(2)}(n) = G_m^{(2)}(n, n)$  and  $G_m^{(2)}(r, n)$  is given by (2.3)–(2.6) with  $i = 2$ . If  $r \geq 4$ , we rewrite (2.10) as

$$x(n) \leq A_3(n) + J_{3m}^*(n; x), \quad n \in N.$$

Here

$$A_3(n) = \prod_{q=1}^2 U^{(q)}(n) \left\{ p(n) + \sum_{i=4}^r J_{im}(n; x) \right\},$$

where  $J_{3m}^*(n; x)$  is obtained from  $J_{3m}(n; x)$  by replacing  $f_{31}(n, s)$  by the function  $f_{31}(n, s) \prod_{q=1}^2 U^{(q)}(n)$ . Applying Theorem 1 once again to the last inequality we get

$$x(n) \leq \prod_{q=1}^3 U^{(q)}(n) \cdot \left\{ p(n) + \sum_{i=4}^r J_{im}(n; x) \right\}, \quad n \in N.$$

Proceeding in this way, we then obtain the desired bound on  $x(n)$  in (2.2).  
Q.E.D.

## 3. MAIN RESULT

In what follows we shall define

$$J_i^{(j)}(n; v) = \sum_{s_1=n_0}^{n-1} f_{i1}^{(j)}(n, s_1) \\ \times \sum_{s_2=n_0}^{s_1-1} f_{i2}^{(j)}(s_1, s_2) \cdots \sum_{s_j=n_0}^{s_{j-1}-1} f_{ij}^{(j)}(s_{j-1}, s_j) v(s_j).$$

The next result deals with a very general form of linear discrete inequalities of the Gronwall type in one independent variable. This is an analogue of an integral inequality established by the present author in [20, Theorem 4].

**THEOREM 3.** *Let  $x(n), p(n)$  be the same as in Theorem 1; let  $f_{ik}^{(j)}(n, s)$  be real-valued nonnegative functions defined on  $N \times N$ , and which are non-decreasing in  $n$  for every  $s \in N$  fixed. Suppose that the discrete inequality*

$$x(n) \leq p(n) + \sum_{j=1}^q \sum_{i=1}^{r_j} J_i^{(j)}(n; x), \quad \text{for } n \in N \quad (3.1)$$

holds, where  $r_j$  are known positive integers. Then we have

$$x(n) \leq p(n) \sum_{j=1}^q \left( \sum_{i=1}^{r_j} B_j^{(i)}(n) \right), \quad n \in N, \quad (3.2)$$

where  $B_j^{(i)}(n) = H_{jj}^{(i)}(n, n)$  and where  $H_{kj}^{(i)}(r, n)$  are defined inductively on the index  $j$  by

$$H_{ij}^{(i)}(r, n) = \prod_{s=n_0}^{n-1} \left\{ 1 + \sum_{k=1}^j F_{ik}^{(j)}(r, s) \right\}, \quad (3.3) \\ H_{kj}^{(i)}(r, n) = \prod_{s=n_0}^{n-1} a_{ij-k+1}^{(j)}(r, s) + \sum_{s=n_0}^{n-1} F_{ij-k+1}^{(j)}(r, s) \\ \times H_{k-1,j}^{(i)}(r, s) \prod_{t=s+1}^{n-1} a_{ij-k+1}^{(j)}(r, t),$$

for  $j = 1, 2, \dots, q$ ;  $i = 1, 2, \dots, r_j$ ;  $k = 2, 3, \dots, j$ , where

$$a_{ih}^{(j)}(r, n) = \begin{cases} c_{ih}^{(j)}(r, n) - F_{ih}^{(j)}(r, n), & \text{if this expression} \geq 0 \text{ on } N \times N, \\ c_{ih}^{(j)}(r, n), & \text{otherwise,} \end{cases} \quad (3.4)$$

$$c_{ih}^{(j)}(r, n) = 1 + \sum_{k=1}^{h-1} F_{ik}^{(j)}(r, n), \quad h = 1, 2, \dots, j-1, \quad (3.5)$$



and

$$F_{il}^{(j)}(n, s) = f_{il}^{(j)}(n, s) \prod_{k=1}^{i-1} B_j^{(k)}(n) \cdot \prod_{h=1}^{j-1} \left( \prod_{m=1}^{r_h} B_h^{(m)}(n) \right), \quad (3.6)$$

$$F_{ik}^{(j)}(n, s) = f_{ik}^{(j)}(n, s), \quad k = 2, 3, \dots, j.$$

*Proof.* The proof can be obtained by using Theorem 2 and an inductive argument. To abridge the argument we point out merely a few steps here. Rewrite (3.1) as

$$x(n) \leq E_1(n) + \sum_{i=1}^{r_1} J_i^{(1)}(n; x), \quad n \in N, \quad (3.1')$$

where

$$E_1(n) = p(n) + \sum_{j=2}^q \sum_{i=1}^{r_j} J_i^{(j)}(n; x).$$

An application of Theorem 2 to (3.1') yields

$$x(n) \leq E_1(n) \prod_{i=1}^{r_1} B_1^{(i)}(n), \quad n \in N, \quad (3.2')$$

where  $B_1^{(i)}(n)$  are given by (3.3)–(3.5) with  $j = 1$ . Rewrite the last inequality as

$$x(n) \leq E_2(n) + \sum_{i=1}^{r_2} \left( \prod_{k=1}^{r_1} B_1^{(k)}(n) \right) J_i^{(2)}(n; x), \quad n \in N$$

$$E_2(n) = \left\{ p(n) + \sum_{j=3}^q \sum_{i=1}^{r_j} J_i^{(j)}(n; x) \right\} \prod_{k=1}^{r_1} B_1^{(k)}(n).$$

Now a suitable application of Theorem 2 to the above inequality gives

$$x(n) \leq E_2(n) \prod_{k=1}^{r_2} B_2^{(k)}(n)$$

$$\leq \prod_{j=1}^2 \left( \prod_{k=1}^{r_j} B_j^{(k)}(n) \right) \left\{ p(n) + \sum_{j=3}^q \sum_{i=1}^{r_j} J_i^{(j)}(n; x) \right\}, \quad n \in N.$$

Continuing in this way we then obtain the desired bound for  $x(n)$  in (3.2).  
Q.E.D.

*Remark 2.* Many particular cases of (3.1) when  $r_j = 1$  hold for  $j = 1, 2, \dots, q$  have been studied by Pachpatte [11, Theorems 1–3] and Agarwal and Thandapani [1, Theorems 1–3]. However, our consequences

for these special cases are not comparable with those known results. Further, the special case of (3.1) when  $r_j = 1, j = 1, 2$ , has been discussed by the present author [19, Theorem 2] under the additional condition such that  $p(n) > 0$  for all  $n > n_0$ .

#### 4. TWO NONLINEAR GENERALIZATIONS

We now establish two nonlinear extensions of Theorem 3 which are useful for some situations.

**THEOREM 4.** *Let all hypotheses of Theorem 3 be satisfied and let  $H: [0, \infty) \rightarrow [0, \infty)$  be strictly increasing and subadditive, with  $H(0) \equiv 0$ . Suppose that the nonlinear discrete inequality*

$$x(n) \leq p(n) + H^{-1} \left\{ \sum_{j=1}^q \sum_{i=1}^{r_j} J_i^{(j)}(n; H(x)) \right\}, \quad n \in N \quad (4.1)$$

*holds, where  $H^{-1}$  denotes the inverse of  $H$ . Then we also have the inequality*

$$x(n) \leq H^{-1} \left\{ H(p(n)) \prod_{j=1}^q \left( \prod_{i=1}^{r_j} B_j^{(i)}(n) \right) \right\}, \quad \text{for } n \in N_1 \subseteq N, \quad (4.2)$$

*where  $B_j^{(i)}(n)$  are the same as defined in Theorem 3, and  $N_1$  is chosen so that*

$$H(p(n)) \prod_{j=1}^q \left( \prod_{i=1}^{r_j} B_j^{(i)}(n) \right) \in \text{Dom}(H^{-1}) \quad \text{when } n \in N_1.$$

*Proof.* We may easily derive from (4.1) that

$$\begin{aligned} y(n) &\leq H \left( p(n) + H^{-1} \left\{ \sum_{j=1}^q \sum_{i=1}^{r_j} J_i^{(j)}(n; y) \right\} \right) \\ &\leq H(p(n)) + \sum_{j=1}^q \sum_{i=1}^{r_j} J_i^{(j)}(n; y), \quad n \in N, \end{aligned} \quad (4.3)$$

herein  $y(n) \equiv H(x(n))$ , since  $H$  is nondecreasing and subadditive. An application of Theorem 3 to (4.3) yields

$$H(x(n)) \leq H(p(n)) \prod_{j=1}^q \left( \prod_{i=1}^{r_j} B_j^{(i)}(n) \right), \quad n \in N, \quad (4.4)$$

where  $B_j^{(i)}(n)$  are the same as in Theorem 3. Now, the desired inequality (4.2) follows from (4.4) immediately, since  $H^{-1}$  is nondecreasing. The choice of  $N_1$  is obvious.

**THEOREM 5.** *Let all hypotheses of Theorem 4 be satisfied. Suppose further that  $H$  is also submultiplicative. If the inequality*

$$x(n) \leq p(n) + g(n) H^{-1} \left\{ \sum_{j=1}^q \sum_{i=1}^{r_j} J_i^{(j)}(n; H(x)) \right\}, \quad n \in N \quad (4.5)$$

*holds, where  $g(n)$  is a real-valued nonnegative function on  $N$ , then for  $n \in N_2 \subseteq N$  we have*

$$x(n) \leq H^{-1} \left\{ H(p(n)) \prod_{j=1}^q \left( \prod_{i=1}^{r_j} \tilde{B}_i^{(j)}(n) \right) \right\}, \quad (4.6)$$

*where  $\tilde{B}_i^{(j)}(n)$  are obtained from  $B_i^{(j)}(n)$  by replacing the functions  $f_H^{(j)}(n, s)$  by  $H(g(n))f_H^{(j)}(n, s)$ , respectively, and  $N_2$  is chosen so that*

$$H(p(n)) \prod_{j=1}^q \left( \prod_{i=1}^{r_j} \tilde{B}_j^{(i)}(n) \right) \in \text{Dom}(H^{-1}) \quad \text{for all } n \in N_2.$$

*Proof.* We observe from (4.5) that

$$H(x(n)) \leq H(p(n)) + H(g(n)) \sum_{j=1}^q \sum_{i=1}^{r_j} J_i^{(j)}(n; H(x)), \quad n \in N$$

since  $H$  is nondecreasing, subadditive, and submultiplicative. The last inequality can be rewritten as

$$y(n) \leq H(p(n)) + \sum_{j=1}^q \sum_{i=1}^{r_j} \tilde{J}_i^{(j)}(n; y), \quad n \in N, \quad (4.7)$$

where  $y(n) \equiv H(x(n))$ , and  $\tilde{J}_i^{(j)}(n; y)$  are obtained from  $J_i^{(j)}(n; y)$  by changing the functions  $f_H^{(j)}(n, s)$  to  $H(g(n))f_H^{(j)}(n; s)$ , respectively. Now, an application of Theorem 3 to (4.7) yields

$$H(x(n)) \leq H(p(n)) \prod_{j=1}^q \left( \prod_{i=1}^{r_j} \tilde{B}_j^{(i)}(n) \right), \quad n \in N, \quad (4.8)$$

where  $\tilde{B}_j^{(i)}(n)$  are as defined in (4.6). Thus, the bound on  $x(n)$  in (4.6) follows from (4.8) immediately, since  $H^{-1}$  is nondecreasing. The choice of  $N_2$  is obvious.

We note that if we set  $H(z) \equiv z$  in Theorems 4 and 5 we obtain Theorem 3. To conclude this paper we notice that we can apply Theorem 3 to extend some results of [19] to contain finite difference equations that involve multiple summations.

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